Charge localization instability in a highly deformable dielectric elastomer

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This paper shows that a highly deformable capacitor made of a soft dielectric and two conformal electrodes can switch between two states discontinuously, by a first-order transition, as the total charge varies gradually. The charged capacitor by itself is a closed thermodynamic system. We adopt the model of ideal dielectric elastomer and assume the Helmholtz free energy density of the form

\[ W = \frac{\mu}{2} (2\lambda^2 + \lambda^{-4} - 3) + \frac{D^2}{2\varepsilon}, \]

where \( \mu \) is the shear modulus and \( \varepsilon \) the (absolute) permittivity, both of which are material constants. The free energy is a sum of two parts: Elastic energy described by the neo-Hookean model and electrostatic energy described by the linear dielectric model, \( D = \varepsilon E \). Furthermore, the model assumes incompressibility of the elastomer, \( HA = ha \), so that \( h = H\lambda^{-2} \). The free energy of the charge-controlled capacitor is \( F_Q = HAW \), namely

\[ F_Q = \frac{\mu HA}{2} (2\lambda^2 + \lambda^{-4} - 3) + \frac{Q^2 H}{2\varepsilon A} \lambda^{-4}. \]

At a fixed charge, the free energy as a function of the stretch, \( F_Q(\lambda) \), has a minimum (Fig. 1(a)). Setting \( dF_Q/d\lambda = 0 \) with \( Q \) held constant, we find the equation of state

\[ Q^2 = \mu e A^2 (\lambda^6 - 1). \]
This charge-stretch relation is monotonic. For a fixed charge, Eq. (3) determines a unique value of stretch corresponding to a stable state of equilibrium. This analysis reproduces another existing result: Voltage-controlled actuation can undergo pull-in instability.22

Pull-in instability in voltage-controlled actuation has been studied for different geometries, such as flat membranes, spherical balloons, tubular balloons,23,28–30 and under different loading conditions, such as equal-biaxial loading, uniaxial loading, and pure-shear conditions.31–33 Pull-in instability often leads to electrical breakdown and should be eliminated in the design of actuators.23 On the other hand, one can design actuators to operate near the verge of instability, leading to safe, giant actuation.34 Voltage-actuated areal expansions over 1000% have been demonstrated.29

Recalling that \( \Phi = \varepsilon H A \), \( h = H \lambda^{-2} \), \( Q = Da \), \( a = \lambda^2 A \), and \( D = \varepsilon E \), one can confirm that the two equations of state, (3) and (5), are identical. We plot the equation of state on the charge-voltage plane using both (3) and (5) by regarding the stretch as a parameter. The charge-voltage curve is not monotonic (Fig. 1(c)). For charge-controlled actuation, charge is gradually added to the homogeneous system (e.g., by corona charging), a single stable state changes continuously, and no instability occurs. The maximum actuation stretch is limited by electrical breakdown field. For voltage-controlled actuation, as the voltage \( \Phi \) ramps up and reaches its maximum value, two branches of solutions merge. No state of equilibrium exists for higher voltages, resulting in electromechanical instability. Setting \( dF_\Phi / d\lambda = 0 \) with \( \Phi \) held constant, we find the equation of state

\[
\varepsilon \Phi^2 = \mu H^2 (\lambda^{-2} - \lambda^{-8}).
\] (5)

This voltage-stretch relation is not monotonic and has a maximum. For a fixed voltage below this maximum, (5) determines two values of stretch, the smaller one corresponding to a stable and the larger one to an unstable state of equilibrium. This analysis reproduces another existing result: Voltage-controlled actuation can undergo pull-in instability.22

We next turn to localization in charge-controlled actuation (Fig. 2(a)). We seek the condition of instability. The entire membrane is electrically connected. Once a small region of the membrane loses stability and becomes thinner than the rest of the membrane, charge will flow to the small region, and the positive feedback will lead to localization. The rest of the membrane will not thin down. Because the small region is constrained by the rest of the membrane, after
charge localization, the small region will have larger stretch in area than the surrounding membrane. Consequently, the small region may form wrinkles. In this paper, we do not analyze the critical condition for the onset of wrinkles, but simply assume that the in-plane stress is zero everywhere in the membrane. For a dielectric sandwiched between two electrodes, even when the total charge on the capacitor is fixed, charge can flow in the electrodes and localize in a small region of the capacitor. In a simplified model, we represent the capacitor by two regions, which are electrically connected and have the same voltage $\Phi$ (Fig. 2(b)). In the reference state, the capacitor is uncharged and undeformed, the two regions have the same thickness $H$, the small region has area $A_S$, the large region has area $A_L$, and the capacitor has the total area $A = A_S + A_L$. In the actuated state, each region can undergo its own homogeneous deformation. They have areas $a_S$ and $a_L$, and the in-plane equal-biaxial stretches are $\lambda_S = a_S/A_S$ and $\lambda_L = a_L/A_L$, respectively. This model neglects the effect of the boundary between the two regions. The charges on the two capacitors are $Q_L = a_L D_L = \varepsilon \lambda_L^2 A_L \Phi / H$ and $Q_S = a_S D_S = \varepsilon \lambda_S^2 A_S \Phi / H$. Once the capacitor receives electric charge, the power source is disconnected, so that the total charge $Q$ on the two regions is fixed and is the sum of the charges on the two regions, $Q = Q_L + Q_S$, namely

$$Q = e \left( \lambda_L^2 A_L + \lambda_S^2 A_S \right) \Phi / H.$$  

(6)

The two regions are two capacitors in parallel, and the coefficient of the charge-voltage relation (6) is their combined capacitance, $C = e \left( \lambda_L^2 A_L + \lambda_S^2 A_S \right) / H$. The free energies per unit volume of the two regions, $W_L$ and $W_S$, are calculated from (1) using quantities of the two regions. The free energy of this composite system is a sum of the free energies of both regions, $F = H A_S W_L + H A_S W_S$, namely

$$F = \frac{\mu H A_L}{2} \left( 2 \lambda_L^2 + \lambda_S^{-2} - 3 \right) + \frac{\mu H A_S}{2} \left( 2 \lambda_S^2 + \lambda_L^{-2} - 3 \right) + \frac{HQ^2}{2 (\lambda_S^2 A_L + \lambda_L^2 A_S)}.$$  

(7)

The first two terms are due to elasticity of the two regions, while the last term is the total electrostatic energy. The free energy of the two-region system is a function of the two stretches $(\lambda_S, \lambda_L)$, and the behavior of this function depends on the value of the total charge $Q$. At a small total charge, the free energy function has only one minimum, corresponding to a homogeneous, stable state of equilibrium (Fig. 3(a)). At an intermediate total charge, two additional extrema (one minimum and one saddle point) appear, corresponding to two inhomogeneous states of equilibrium (Fig. 3(b)). At a large total charge, the homogeneous state becomes a saddle point, and the system will stabilize at an inhomogeneous state (Fig. 3(c)).
Setting $\partial F(\lambda_L, \lambda_S)/\partial \lambda_L = \partial F(\lambda_L, \lambda_S)/\partial \lambda_S = 0$ in (7) with the total charge $Q$ held constant, we obtain two equations of state

$$\varepsilon \Phi^2 = \mu H^2 \left( \lambda_L^2 - \lambda_L^8 \right),$$

$$\varepsilon \Phi^2 = \mu H^2 \left( \lambda_S^2 - \lambda_S^8 \right).$$

At a given total charge $Q$, (6), (8), and (9) form a set of nonlinear equations for three unknowns: $\lambda_L$, $\lambda_S$, and $\Phi$. Each solution corresponds to a state of equilibrium of the two-region system. Because the equations are nonlinear, a capacitor subject to a given total charge may have multiple states of equilibrium. We regard the total charge as the control parameter and plot these states of equilibrium in bifurcation diagrams: On the charge-voltage plane (Fig. 3(d)), the charge-$\lambda_S$ plane (Fig. 3(e)), and the charge-$\lambda_L$ plane (Fig. 3(f)). As the voltage is the same for both regions, (8) and (9) always have a trivial solution $\lambda_S = \lambda_L$, corresponding to a homogeneous state (black curves in Figs. 3(d)–3(f)). Two branches of inhomogeneous solutions exist, corresponding to $\lambda_S < \lambda_L$ (red curves) and $\lambda_S > \lambda_L$ (blue curves). To show the branches clearly, we use different scales for the ordinates in Figs. 3(e) and 3(f).

The multiplicity of solutions can be understood graphically. The voltage $\Phi$ is common for the two parallel capacitors, and the total charge is the sum $Q = Q_L + Q_S$. The charge-voltage relations for the two capacitors have the form $Q_L = A_L f(\Phi / H_L)$ and $Q_S = A_S f(\Phi / H_S)$, where the universal function $f$ is the same as that for the homogeneous system (Fig. 1(c)). For a given voltage below the maximum, each of the functions $Q_L(\Phi)$ and $Q_S(\Phi)$ gives two values of charge. The curves in Fig. 3(d) can be obtained by adding the charge-voltage curves of the two capacitors, $Q(\Phi) = Q_L(\Phi) + Q_S(\Phi)$. Thus, the resulting function $Q(\Phi)$ has four values of charge for each value of voltage. Furthermore, for the two capacitors of the same initial thickness, $H_L = H_S = H$, the maximum voltage for the two capacitors is the same as that for the homogeneous system, $\Phi_c$. The total charge at this voltage is the sum $Q_L + Q_S = (A_L + A_S) f(\Phi_c / H) = A f(\Phi_c / H)$, which is the same as the critical charge of the homogeneous system, $Q_c$. Thus, all four branches of solutions merge at the single-capacitor critical values $(Q_c, \Phi_c)$.

Imagine that we ramp up the total charge on the capacitor gradually. When the total charge is below the critical value, $Q < Q_c$, the two regions deform by the same stretch, along the curve of homogeneous deformation. When the total charge exceeds the critical value, $Q > Q_c$, the homogeneous deformation becomes unstable, and both branches of inhomogeneous states are stable. Consequently, the capacitor must switch from the homogeneous state to one of the inhomogeneous states. At the fixed charge $Q = Q_c$, the vertical purple lines represent the capacitor that snaps from the homogeneous state to a state with localized charge (blue curves). This localized state has a lower voltage (Fig. 3(d)), an expanded small region (Fig. 3(e)), and a contracted large region (Fig. 3(f)). The switch from the homogeneous state to the localized state is a first-order transition and greatly reduces the free energy of the capacitor. At $Q = Q_c$, the capacitor can also make another transition, without discontinuity in voltage or stretches of the two regions (red curves), of lower voltage, contracted small region and expanded large region. This continuous transition reduces free energy only by a small amount and is therefore less favorable. We next focus on the snapping transition.

In the course of the snapping transition, charges flow from the large region to the small region, and this localization amplifies the electrical field in the small region, which may lead to electrical breakdown. We plot another bifurcation diagram in the plane of the fixed $Q$ and the electric field in the small region $E_S$, where we only include the stable region of the homogeneous branch (black) and the blue branch of the inhomogeneous state (Fig. 4(a)). Each blue curve corresponds to a given value of $A_S/A$, the vertical purple line corresponds to the snap, and the intersection between the blue curve and the purple line determines the electric field in the small region after the snap. The snap amplifies the electric field more if the relative size of the small region $A_S/A$ is smaller.

Assume that the breakdown electric field $E_{EB}$ is large enough, so that the dielectric does not suffer electrical breakdown in the homogeneous state, $E_{EB} > E_c$, namely, $E_{EB} > 2^{-2/3} \sqrt{3} \mu / \varepsilon$. After the capacitor snaps into the inhomogeneous state, the electric field in the small region amplifies to $E_S$. This snap will cause electrical breakdown if $E_S > E_{EB}$. The electric field $E_S$ is a decreasing function of $A_S/A$. Consequently, whether the electromechanical localization will cause electrical breakdown depends on a combination of material properties, $E_{EB} \sqrt{\varepsilon / \mu}$, and the relative size of the snapping region, $A_S/A$. The two parameters form a

![FIG. 4. Electromechanical localization and electrical breakdown. (a) Bifurcation diagram plotted on the plane of the total charge $Q$ and the electric field in the small part $E_S$. (b) The condition under which electromechanical localization does not cause electrical breakdown.](image-url)
plane, in which the condition \( E_S = E_{EE} \) is a curve (Fig. 4(b)). Above this curve, the localization will not cause electrical breakdown. Below this curve, the localization will cause electrical breakdown.

We derive the asymptotic behavior of the localized state in the limit of small \( A_S/A \). In this limit, the small region deforms greatly, \( \lambda_S \gg 1 \), while the large region is nearly undeformed, \( \lambda_L \to 1 \). Consequently, (8) and (9) give that \( \lambda_L \approx 1 + \lambda_S^{-2}H/6 \) and \( \Phi \approx H \sqrt{\mu/\lambda_S^2} \). Substituting these two expressions into (6), replacing \( Q \) by \( Q_l = \sqrt{3\epsilon\mu}A \), and retaining the leading term, we obtain \( \lambda_S \approx 3^{1/6}(A_S/A)^{-1/3} \). The electric field in the small region is \( E_S = \lambda_S^2\Phi/H \approx 3^{1/6} \sqrt{\mu/\epsilon}(A_S/A)^{-1/3} \). This power law closely approximates the numerical solution for small \( A_S/A \) (Fig. 4(b)).

In this limit, the charge remaining in the large region is a small fraction of the total charge on the capacitor, \( Q_l/Q \approx 3^{-2/3}(A_S/A)^{1/3} \).

We are unaware of any direct experimental observation of charge localization. However, the instability of a homogeneous dielectric membrane has been predicted using a similar theoretical procedure, and has been verified by experiments. The electromechanical localization is reminiscent of a well-known mechanical instability. When a metallic wire is pulled beyond a certain strain, homogeneous deformation becomes unstable, and the wire forms a neck. This necking instability will set in even when the wire is in displacement-controlled tension. The neck will lead to fracture if the wire is long, but will stabilize if the wire is short. In the necking instability of long metal wires, the length of the neck is comparable to the diameter of the wire. Similarly, we expect that the electromechanical localization will occur over an area about \( A_S \approx H^2 \).

For a commonly used dielectric elastomer VHB™, the representative value is \( E_{EE} \sqrt{\epsilon/\mu} = 4.73 \). The localized state does not undergo electrical breakdown if \( A_S/A > 0.016 \). This condition translates to the area of the electrode below about \( A \approx 61H^2 \), or the diameter of the electrode below \( D \approx 8H \). For an initially 0.5 mm thick membrane, for example, the corresponding “breakdown-safe” linear size will be in the mm range. If the breakdown does not occur, the capacitor can be switched between the homogeneous and the localized state repeatedly. Upon this switch, the voltage drops significantly, and the small region deforms greatly. These characteristics enable devices with bistable states, such as Braille displays. Furthermore, the bistable states can be tuned and modified in many ways, such as by using a stiffening elastomer with a relatively small limiting stretch, applying a prestretch, introducing imperfections, and laminating the deformable capacitor (or part of it) with a passive soft layer.

In summary, when the electric charge is immobile on the surfaces of the dielectric, charge-controlled actuation is stable and does not suffer pull-in instability. In the presence of electrodes, however, charges will be mobile, and charge-controlled actuation is bistable. At a critical charge, the homogeneous deformation becomes unstable, and the capacitor will snap into a state of localized deformation by a first-order transition, which may lead to electrical breakdown. However, the breakdown in the charge-localization region can be avoided if the initial area of the electrodes is small. This bistability is tunable and can be used to design devices.

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